The Jordan 1-Structure of a Matrix of Redheffer

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ABSTRACT

Let $\epsilon_1 = (1,0,\dots,0)$ and $\epsilon = (1,1,\dots,1)$ be complex vectors of length n, and let $D_n = (d_{ij})$, with $d_{ij} = 1$ if $i \mid j$ and 0 otherwise, be an $n \times n$ complex matrix. Then the sizes of the Jordan blocks of the matrix $\epsilon^{\mathrm{T}}\epsilon_1 + D_n$ corresponding to the eigenvalue 1 are

$$[\log_2(n/3)]+1, [\log_2(n/5)]+1, \dots, [\log_2(n/\{n\})]+1,$$

where $\{n\}$ is the greatest odd integer less than or equal to n.

INTRODUCTION

Let $\epsilon_1 = (1,0,\dots,0)$ and $\gamma_n = (c_1, c_2, \dots, c_n)$ be vectors in the space $\mathbb{C}^n$ of rows of length n over the field $\mathbb{C}$ of complex numbers. Let $C_n = \gamma_n^{\mathrm{T}}\epsilon_1$, $D_n = (d_{ij})$ with $d_{ij} = 1$ if $i \mid j$ and 0 otherwise, and let $A_n = C_n + D_n$ be $n \times n$ complex matrices. For example,

$$A_6 = \begin{pmatrix} c_1+1 & 1 & 1 & 1 & 1 & 1 \\ c_2 & 1 & 0 & 1 & 0 & 1 \\ c_3 & 0 & 1 & 0 & 0 & 1 \\ c_4 & 0 & 0 & 1 & 0 & 0 \\ c_5 & 0 & 0 & 0 & 1 & 0 \\ c_6 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

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The matrix A_n was introduced by R. Redheffer in [4]. It is known that the determinant of A_n is $1+\sum_{k=1}^{n} c_k \mu(k)$, where μ is the Möbius function. In particular, if $\gamma_n = (0,1,\ldots,1)$, $n > 1$, then the determinant of A_n is Mertens's function $M(n) = \sum_{k=1}^{n} \mu(k)$. (See [1] or [4].) Consequently, in this case, $|\det A_n| = O(n^{1/2+\epsilon})$ for every $\epsilon > 0$ if and only if the Riemann hypothesis is true. (See, for example, [5].)

We proceed by introducing some notation. First, the greatest integer $[x]$ of a real number x is the unique integer satisfying

$$[x] \leqslant x < [x]+1. \tag{0.1}$$

Second, for convenience we often let

$$\nu(x) = [\log_2 x]. \tag{0.2}$$

That is, if x is a positive real number, then $\nu(x)$ is the unique integer for which $2^{\nu(x)} \leqslant x < 2^{\nu(x)+1}$. Third, for $\gamma_n = (c_1, c_2, \ldots, c_n)$ in $\mathbf{C}^n$, we define

$$\tau(\gamma_n) = \begin{cases} c_{2^{\nu(n)}} & \text{if} \quad n < 3\times 2^{\nu(n)-1} \\ c_{2^{\nu(n)}} + \nu(n) c_{3\times 2^{\nu(n)-1}} & \text{if} \quad n \geqslant 3\times 2^{\nu(n)-1}. \end{cases} \tag{0.3}$$

We note that $3\times 2^{\nu(n)-1}$ is the "midpoint" of the "interval" $[2^{\nu(n)}, 2^{\nu(n)+1})$ containing n. Thus, for example $\nu(4) = \nu(5) = \nu(6) = \nu(7) = 2$, and $\tau(\gamma_4) = \tau(\gamma_5) = c_4$, while $\tau(\gamma_6) = \tau(\gamma_7) = c_4 + 2c_6$.

It was recently shown [1] for the special case $\gamma_n = (0,1,\ldots,1)$ with $n > 1$ that 1 is an eigenvalue of A_n of multiplicity $n - \nu(n) - 1$ in the characteristic polynomial of A_n. In the present paper we characterize those A_n for which this result holds. Specifically, let $\sigma_1(n)$ be the multiplicity of 1 in the characteristic polynomial of A_n. We show below that $\sigma_1(n) = n - \nu(n) - 1$ iff $\tau(\gamma_n) \neq 0$; futhermore, in this case, A_n is similar to a direct sum of a cyclic matrix of order $\nu(n)+1$, whose minimum polynomial $h_n(x)$ satisfies $h_n(1) \neq 0$, and $[(n-1)/2]$ Jordan blocks associated with the eigenvalue 1 of orders $\nu(n/k)+1$, $k = 3,5,\ldots,2[(n-1)/2]+1$.

For example, let $n = 6$. Then $6 - \nu(6) - 1 = 3$ and $\tau(\gamma_6) = c_4 + 2c_6$. By direct calculation the characteristic polynomial of A_6 is $(x-1)^3 h_6(x)$, where

$$h_6(x) = x^3 - (c_1+3)x^2 + \left(3(c_1+1) - \sum_{k=1}^{6} c_k\right)x - \left((c_4+2c_6) + 2c_1 + 1 - \sum_{k=1}^{6} c_k\right).$$

Since $h_6(1) = -(c_4 + 2c_6)$, then it follows that the multiplicity $\sigma_1(6)$ of the root 1 is 3 iff $\tau(\gamma_6) \neq 0$. In this case A_6 is similar to the direct sum of a cyclic matrix with minimum polynomial $h_6(x)$ of degree $3 = \nu(6)+1$ and $2 = [(6-1)/2]$ Jordan 1-blocks,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad (1),$$

of orders $2 = \nu(6/3)+1$ and $1 = \nu(6/5)+1$.

THE MAIN RESULTS

We begin with an investigation of the strictly upper triangular part T_n of the matrix $D_n = (d_{ij})$, where $d_{ij} = 1$ if $i \mid j$ and 0 otherwise.

First, given integers i, j, and m with $1 \leqslant i,\ j \leqslant n$ and $m \geqslant 0$, we say that an $(i, j; m)$-list is a list $(h_1, h_2, \ldots, h_m)$ of m integers greater than 1 such that $j = ih_1h_2 \cdots h_m$. We demonstrate the following fact.

Lemma 1. *Let $T_n = D_n - I_n$, and let $T_n^m = (t_{ij}^{(m)})$ with $1 \leqslant i,\ j \leqslant n$ and $m \geqslant 0$. Then $t_{ij}^{(m)}$ is the number of $(i, j; m)$-lists.*

Proof. Since

$$t_{ij} = t_{ij}^{(1)} = \begin{cases} 1 & \text{if} \quad i \mid j \text{ and } i < j \\ 0 & \text{otherwise,} \end{cases}$$

then $t_{ik_1}t_{k_1k_2} \cdots t_{k_{m-1}j} = 1$ if and only if

$$i \mid k_1, \quad i < k_1, \qquad k_1 \mid k_2, \quad k_1 < k_2, \ldots, \qquad k_{m-1} \mid j, \quad k_{m-1} < j.$$

Consequently, since

$$t_{ij}^{(m)} = \sum_{k_1, \ldots, k_{m-1} = 1}^{n} t_{ik_1}t_{k_1k_2} \cdots t_{k_{m-1}j},$$

then $t_{ij}^{(m)}$ is the number of lists $(i, k_1, k_2, \ldots, k_{m-1}, j)$ of integers such that $i \mid k_1 \mid k_2 \mid \cdots \mid k_{m-1} \mid j$ with $1 \leqslant i < k_1 < k_2 < \cdots k_{m-1} < j \leqslant n$. But, from the following observation, such lists are in one-to-one correspondence with

$(i, j; m)$-lists: if $i \mid k_1 \mid k_2 \mid \cdots \mid k_{m-1} \mid j$ with $i < k_1 < \cdots < k_{m-1} < j$, then

$$j = i \frac{k_1}{i} \frac{k_2}{k_1} \cdots \frac{k_{m-1}}{k_{m-2}} \frac{j}{k_{m-1}}$$

and $(k_1/i, k_2/k_1, \ldots, k_{m-1}/k_{m-2}, j/k_{m-1})$ is an $(i, j; m)$-list; conversely, if $(h_1, h_2, \ldots, h_m)$ is an $(i, j; m)$-list, then $j = ih_1h_2 \cdots h_m$ and

$$i \mid ih_1 \mid ih_1h_2 \mid \cdots \mid ih_1h_2 \cdots h_m$$

with $1 \leqslant i < ih_1 < ih_1h_2 < \cdots < ih_1h_2 \cdots h_m < j \leqslant n$. ∎

We illustrate Lemma 1 with the example $n = 6$. In this case,

$$T_6 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$T_6^2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $T_6^3 = 0$. In particular, since $4 = 1 \times 2 \times 2$, then clearly the pair $(2,2)$ is the one and only $(1,4;2)$-list and $t_{1,4}^{(2)} = 1$; since $6 = 1 \times 2 \times 3 = 1 \times 3 \times 2$, then there are two $(1,6;2)$-lists and $t_{1,6}^{(2)} = 2$. Finally, if $j = ih_1h_2$ for $1 \leqslant i,\ j \leqslant 6$ and integers h_1, h_2 greater than 1, then necessarily $i = 1$ and $j = 4$ or 6. Thus, every other entry of T_6^2 is zero.

We now summarize the details about T_n that are used in this paper.

COROLLARY 1. *Let $t_{ij}^{(m)}$ be given as in Lemma 1. Then $t_{ij}^{(m)} = 0$ whenever*

(1.1) $j < i \times 2^m$,
(1.2) $i > [n/2^m]$,
(1.3) $m \geqslant [\log_2(n/i)] + 1$,
(1.4) $m \geqslant [\log_2 n] + 1$,

(1.5) $0 \leqslant m < [\log_2 n]$, $i = [n/2^m]$, $j > i \times 2^m$,
(1.6) $m = [\log_2 n]$, $2^m < j \leqslant n < 3 \times 2^{m-1}$,
(1.7) $m = [\log_2 n]$, $2^m < j \neq 3 \times 2^{m-1} \leqslant n$.

Moreover, if $1 \leqslant i \leqslant [n/2^m]$, *then* $t^{(m)}_{i,i\times 2^m} = 1$; *and if* $m = [\log_2 n]$ *with* $3 \times 2^{m-1} \leqslant n$, *then* $t^{(m)}_{1,3\times 2^{m-1}} = m$.

Proof. (1.1): If $j < i \times 2^m$, then there are no $(i, j; m)$-lists.

(1.2): If $i > [n/2^m]$, then $n/2^m < i$, so that $j \leqslant n < i \times 2^m$ and $t^{(m)}_{ij} = 0$ by (1.1).

(1.3): Let $m \geqslant [\log_2(n/i)] + 1$. Then $m > \log_2(n/i)$. So $n/i < 2^m$ and $j \leqslant n < i \times 2^m$. Again, by (1.1), $t^{(m)}_{ij} = 0$.

(1.4): Since $1 \leqslant i$, this is a consequence of (1.3).

(1.5): Let $0 \leqslant m < [\log_2 n]$, $i = [n/2^m]$, and $j > i \times 2^m$. Then $m + 1 \leqslant [\log_2 n] \leqslant \log_2 n$, $2^{m+1} \leqslant n$, and $2 \leqslant n/2^m$. But, whenever $x \geqslant 2$, then $2x < 3[x]$; in particular, for $x = n/2^m$, $2(n/2^m) < 3[n/2^m]$. Hence, $i \times 2^m < j \leqslant n < i \times 2^{m-1} \times 3$ and there are no $(i, j; m)$-lists.

(1.6): Let $m = [\log_2 n]$ and $2^m < j \leqslant n < 3 \times 2^{m-1}$. Since $[n/2^m] = 1$, then by (1.2) we need only consider the case $i = 1$. Since $i \times 2^m < j \leqslant n < i \times 2^{m-1} \times 3$, there are no $(i, j; m)$-lists.

(1.7): Let $m = [\log_2 n]$ and $2^m < j \neq 3 \times 2^{m-1} \leqslant n$. Again $[n/2^m] = 1$. Since $2^{m-2} \times 3^2 > 2^{m-2} \times 2^3 = 2^{m+1} > n$ and $2^{m-1} \times 5 > 2^{m-1} \times 2^2 = 2^{m+1} > n$, then $t^{(m)}_{ij} = 0$ whenever $j \neq 2^m$ and $j \neq 3 \times 2^{m-1}$.

Next, let $1 \leqslant i \leqslant [n/2^m]$ and $j = i \times 2^m$. Then $1 \leqslant i$, $j \leqslant n$ and there is precisely one $(i, j; m)$-list; that is, $t^{(m)}_{ij} = 1$. Finally, let $m = [\log_2 n]$ with $3 \times 2^{m-1} \leqslant n$. Then, for $i = 1$ and $j = 3 \times 2^{m-1}$, there are m $(i, j; m)$-lists

$$(3,2,2,\dots,2), \quad (2,3,2,\dots,2),\dots, \quad (2,2,2,\dots,3);$$

consequently, in this case, $t^{(m)}_{ij} = m$. ∎

In particular, by (1.4) and the last part of Corollary 1, T_n is nilpotent of index $[\log_2 n] + 1 = \nu(n) + 1$.

We illustrate Corollary 1 with the example $n = 6$. In this case, $\nu(6) = [\log_2 6] = 2$ and $6 \geqslant 3 \times 2^{\nu(6)-1}$; in particular, $t^{(2)}_{1,4} = 1$, $t^{(2)}_{1,6} = [\log_2 6]$, and $T_6^{\nu(6)+1} = 0$.

Lemma 2. *Let* $\epsilon_1 = (1,0,\dots,0)$, $\epsilon_2 = (0,1,\dots,0),\dots,\epsilon_n = (0,0,\dots,1)$, *and* $\gamma_n = (c_1, c_2, \dots, c_n)$ *be row vectors of length* n. *Let* $C_n = \gamma_n^{\mathrm{T}}\epsilon_1$, $A_n = C_n + D_n$, *and* $T_n = D_n - I_n$ *be as above.*

(2.1) *The space of column vectors spanned by*

$$\gamma_n^{\mathsf{T}}, (C_n + T_n)\gamma_n^{\mathsf{T}}, \dots, (C_n + T_n)^m \gamma_n^{\mathsf{T}}$$

is the space spanned by

$$\gamma_n^{\mathsf{T}}, T_n\gamma_n^{\mathsf{T}}, \dots, T_n^m \gamma_n^{\mathsf{T}}.$$

(2.2) *The space of row vectors spanned by*

$$\epsilon_1, \epsilon_1 A_n, \dots, \epsilon_1 A_n^m$$

is the space spanned by

$$\epsilon_1, \epsilon_1 T_n, \dots, \epsilon_1 T_n^m.$$

(2.3) *If* $T_n^{\nu(n)}\gamma_n^{\mathsf{T}} \neq 0$ *and* $m \leqslant \nu(n)+1$, *then the list of column vectors*

$$\gamma_n^{\mathsf{T}}, T_n\gamma_n^{\mathsf{T}}, \dots, T_n^{m-1}\gamma_n^{\mathsf{T}}, \epsilon_1^{\mathsf{T}}, \dots, \epsilon_{[n/2^m]}^{\mathsf{T}}$$

is linearly independent.

(2.4) *If* $m \leqslant \nu(n)+1$, *then the list of row vectors*

$$\epsilon_1, \epsilon_1 T_n, \dots, \epsilon_1 T_n^{m-1}, \epsilon_1 T_n^m, \epsilon_2 T_n^m, \dots, \epsilon_{[n/2^m]} T_n^m$$

is linearly independent.

Proof. (2.1): Since the first vector in both lists is γ_n^{T}, we proceed by induction on m. Thus, we assume the conclusion for a given m, and consider lists of length $m+1$. With the convention that $f_m(T_n)\gamma_n^{\mathsf{T}}$ represents a linear combination of $\gamma_n^{\mathsf{T}}, T_n\gamma_n^{\mathsf{T}}, \dots, T_n^m\gamma_n^{\mathsf{T}}$, then, by use of the fact that $\epsilon_1(C_n + T_n)^m\gamma_n^{\mathsf{T}}$ is a 1×1 scalar matrix,

$$\begin{aligned}
(C_n + T_n)^{m+1}\gamma_n^{\mathsf{T}} &= (C_n + T_n)(C_n + T_n)^m\gamma_n^{\mathsf{T}} \\
&= C_n(C_n + T_n)^m\gamma_n^{\mathsf{T}} + T_n(C_n + T_n)^m\gamma_n^{\mathsf{T}} \\
&= (\gamma_n^{\mathsf{T}}\epsilon_1)(C_n + T_n)^m\gamma_n^{\mathsf{T}} + T_n f_m(T_n)\gamma_n^{\mathsf{T}} \\
&= \left(\epsilon_1(C_n + T_n)^m\gamma_n^{\mathsf{T}}\right)\gamma_n^{\mathsf{T}} + f_{m+1}(T_n)\gamma_n^{\mathsf{T}}
\end{aligned}$$

is in the span of $\gamma_n^{\mathsf{T}}, T_n\gamma_n^{\mathsf{T}}, \dots, T_n^{m+1}\gamma_n^{\mathsf{T}}$. Consequently, the space spanned by the first list is contained in the space spanned by the second. A similar argument provides the reverse containment.

(2.2): By use of the identity $A_n = C_n + I_n + T_n$, an analogous argument may be used to prove (2.2).

(2.3): First, consider the case $m = \nu(n)+1$. Then $[n/2^m] = 0$, and it is understood that the latter part of the list given in (2.3) is empty. By hypothesis, $T_n^{m-1}\gamma_n^{\mathsf{T}} \neq 0$; consequenty, since T_n is nilpotent, then the list $\gamma_n^{\mathsf{T}}, T_n\gamma_n^{\mathsf{T}}, \dots, T_n^{m-1}\gamma_n^{\mathsf{T}}$ is linearly independent.

Next, let $m \leqslant \nu(n)$ and suppose that

$$0 = a_0\gamma_n^{\mathsf{T}} + a_1T_n\gamma_n^{\mathsf{T}} + \cdots + a_{m-1}T_n^{m-1}\gamma_n^{\mathsf{T}} + b_1\epsilon_1^{\mathsf{T}} + \cdots + b_{[n/2^m]}\epsilon_{[n/2^m]}^{\mathsf{T}}.$$

Since $1 \leqslant [n/2^m] \leqslant n/2^m < 2^{\nu(n)-m+1}$, then by (1.1) of Corollary 1 with $i = 1$,

$$T_n^{\nu(n)-m+1}\epsilon_j^{\mathsf{T}} = 0 \qquad \text{for} \quad j = 1, \dots, [n/2^m].$$

Hence,

$$0 = a_0T_n^{\nu(n)-m+1}\gamma_n^{\mathsf{T}} + a_1T_n^{\nu(n)-m+2}\gamma_n^{\mathsf{T}} + \cdots + a_{m-1}T_n^{\nu(n)}\gamma_n^{\mathsf{T}}.$$

Since by hypothesis $T_n^{\nu(n)}\gamma_n^{\mathsf{T}} \neq 0$ and T_n is nilpotent, then $a_0 = 0$, $a_1 = 0, \dots, a_{m-1} = 0$; hence also $b_1 = 0$, $b_2 = 0, \dots, b_{[n/2^m]} = 0$. That is, the list of (2.3) is linearly independent.

(2.4): Let $m = \nu(n)+1$. Then $[n/2^m] = 0$ and the latter part of the list (2.4) past $\epsilon_1T_n^{m-1}$ is empty. Since by Corollary 1, $t_{1,2^{m-1}}^{(m-1)} = 1$, then $\epsilon_1T_n^{m-1} \neq 0$. Consequently, the list $\epsilon_1, \epsilon_1T_n, \dots, \epsilon_1T_n^{m-1}$ is linearly independent. Finally, let $m \leqslant \nu(n)$ and suppose

$$0 = a_0\epsilon_1 + a_1\epsilon_1T_n + \cdots + a_{m-1}\epsilon_1T_n^{m-1} + b_1\epsilon_1T_n^m + \cdots + b_{[n/2^m]}\epsilon_{[n/2^m]}T_n^m.$$

Since T_n is nilpotent and $\epsilon_1T_n^{\nu(n)} \neq 0$, then $a_0 = 0$, $a_1 = 0, \dots, a_{m-1} = 0$; therefore, $\sum_{j=1}^{[n/2^m]} b_j\epsilon_jT_n^m = 0$. But, by Corollary 1, the lead nonzero element of the ith row of T_n^m is a 1 in column $i \times 2^m$, $1 \leqslant i \leqslant [n/2^m]$; in particular, the list $\epsilon_1T_n^m, \dots, \epsilon_{[n/2^m]}T_n^m$ is linearly independent and $b_1 = 0, \dots, b_{[n/2^m]} = 0$. ∎

Theorem 1. *Let $A_n = C_n + D_n$ with $C_n = \gamma_n^{\mathsf{T}}\epsilon_1$ and $D_n = I_n + T_n$ be given as above, and let m be a nonnegative integer. Suppose that $T_n^{\nu(n)}\gamma_n^{\mathsf{T}} \neq 0$.*

Then

$$\operatorname{rank}(A_n - I_n)^m = \begin{cases} m + [n/2^m] & \text{if } m \leqslant \nu(n), \\ \nu(n)+1 & \text{if } m \geqslant \nu(n). \end{cases}$$

Furthermore, if $s_1(n)$ *is the least* m *such that* $\operatorname{rank}(A_n - I_n)^{m+1} = \operatorname{rank}(A_n - I_n)^m$, *then*

$$s_1(n) = \nu(n/3)+1 = \begin{cases} \nu(n)-1 & \text{if } n < 3\times 2^{\nu(n)-1} \\ \nu(n) & \text{if } n \geqslant 3\times 2^{\nu(n)-1}. \end{cases}$$

In particular, $\operatorname{rank}(A_n - I_n)^{s_1(n)} = \nu(n)+1$.

Proof. First, we show by induction on n that

$$(A_n - I_n)^m = \sum_{k=0}^{m-1} (C_n + T_n)^k C_n T_n^{m-1-k} + T_n^m.$$

Indeed, since

$$(A_n - I_n)^1 = C_n + T_n = (C_n + T_n)^0 C_n T_n^0 + T_n^1,$$

we assume the result for the case m and observe that

$$\begin{aligned}(A_n - I_n)^{m+1} &= (A_n - I_n)(A_n - I_n)^m = (C_n + T_n)(A_n - I_n)^m \\ &= \sum_{k=0}^{m-1} (C_n + T_n)^{k+1} C_n T_n^{m-1-k} + (C_n + T_n) T_n^m \\ &= \sum_{k=1}^{m} (C_n + T_n)^k C_n T_n^{m-k} + (C_n + T_n)^0 C_n T_n^m + T_n^{m+1} \\ &= \sum_{k=0}^{(m+1)-1} (C_n + T_n)^k C_n T_n^{(m+1)-1-k} + T_n^{m+1}.\end{aligned}$$

Now, let $m \leqslant \nu(n)+1$. If $m = \nu(n)+1$ then $T_n^m = 0$, and if $m \leqslant \nu(n)$, then by (1.2) of Corollary 1,

$$\left(\epsilon_1^{\mathsf T}\epsilon_1 + \cdots + \epsilon_{[n/2^m]}^{\mathsf T}\epsilon_{[n/2^m]}\right) T_n^m = T_n^m.$$

Consequently, since $C_n = \gamma_n^{\mathsf{T}}\epsilon_1$,

$$(A_n - I_n)^m = \sum_{k=0}^{m-1} (C_n + T_n)^k \gamma_n^{\mathsf{T}}\epsilon_1 T_n^{m-1-k} + \sum_{k=1}^{[n/2^m]} \epsilon_k^{\mathsf{T}}\epsilon_k T_n^m,$$

which is the product of the $n \times (m + [n/2^m])$ matrix F whose columns are

$$(C_n + T_n)^0\gamma_n^{\mathsf{T}},\dots,(C_n + T_n)^{m-1}\gamma_n^{\mathsf{T}},\epsilon_1^{\mathsf{T}},\dots,\epsilon_{[n/2^m]}^{\mathsf{T}}$$

with the $(m + [n/2^m]) \times n$ matrix G whose rows are

$$\epsilon_1 T_n^{m-1},\dots,\epsilon_1,\epsilon_1 T_n^m,\dots,\epsilon_{[n/2^m]} T_n^m.$$

By (2.1) of Lemma 2, the columns of the first factor span the same space as spanned by the columns

$$\gamma_n^{\mathsf{T}}, T_n\gamma_n^{\mathsf{T}},\dots,T_n^{m-1}\gamma_n^{\mathsf{T}},\epsilon_1^{\mathsf{T}},\dots,\epsilon_{[n/2^m]}^{\mathsf{T}}.$$

Consequently, since by hypothesis $T_n^{\nu(n)}\gamma_n^{\mathsf{T}} \neq 0$, by (2.3) and (2.4) of Lemma 2, FG is a full-rank factorization of $(A_n - I_n)^m$; in other words, $(A_n - I_n)^m$ is of rank $m + [n/2^m]$ for $m \leqslant \nu(n)+1$. In particular, since

$$\operatorname{rank}(A_n - I_n)^{\nu(n)+1} = \nu(n) + 1 = \operatorname{rank}(A_n - I_n)^{\nu(n)},$$

it follows that $\operatorname{rank}(A_n - I_n)^m = \nu(n)+1$ for $m \geqslant \nu(n)$.

Now, let $s_1(n)$ be the least nonnegative integer m such that rank $(A_n - I_n)^{m+1} = \operatorname{rank}(A_n - I_n)^m$. Since the rank stabilizes at the first instance of equality, then, in particular, $s_1(n) \leqslant \nu(n)$ and $\operatorname{rank}(A_n - I_n)^{s_1(n)} = \nu(n)+1$. Furthermore, since

$$(m+1) + \left[n/2^{m+1}\right] = m + \left[n/2^m\right] \quad \text{iff} \quad 1 + \left[n/2^{m+1}\right] = \left[n/2^m\right],$$

then $s_1(n)$ is the least nonnegative integer m such that $[n/2^m] - [n/2^{m+1}] = 1$. Now, $[n/2^{\nu(n)+1}] = 0$, $[n/2^{\nu(n)}] = 1$; and if $n < 3\times 2^{\nu(n)-1}$ then $[n/2^{\nu(n)-1}] = 2$, $[n/2^{\nu(n)-2}] \geqslant 4$; and if $n \geqslant 3\times 2^{\nu(n)-1}$, then $[n/2^{\nu(n)-1}] = 3$, $[n/2^{\nu(n)-2}] \geqslant 6$. Consequently, $s_1(n) = \nu(n) - 1$ if $n < 3\times 2^{\nu(n)-1}$, and $s_1(n) = \nu(n)$ if $n \geqslant 3\times 2^{\nu(n)-1}$. Furthermore, in the first case, since $2^{\nu(n)} \leqslant n$

$< 3\times 2^{\nu(n)-1}$, then $2^{\nu(n)-2} < \frac{4}{3}2^{\nu(n)-2} = \frac{1}{3}2^{\nu(n)} \leqslant n/3 < 2^{\nu(n)-1}$ and $\nu(n/3) = \nu(n) - 2 = s_1(n) - 1$; that is, $s_1(n) = \nu(n/3) + 1$. In the second case, $2^{\nu(n)-1} \leqslant n/3 < \frac{1}{3}2^{\nu(n)+1} < 2^{\nu(n)}$, and $\nu(n/3) = \nu(n) - 1 = s_1(n) - 1$; again, $s_1(n) = \nu(n/3) + 1$. ■

Since $[n/2^{\nu(n)}] = 1$, then by the observation above, $\epsilon_1^{\mathsf{T}}\epsilon_1 T_n^{\nu(n)} = T_n^{\nu(n)}$. But by (1.6) and (1.7) of Corollary 1 and the definition of $\tau(\gamma_n)$, as specified in (0.3) of the introduction,

$$\tau(\gamma_n) = \epsilon_1 T_n^{\nu(n)}\gamma_n^{\mathsf{T}}.$$

Therefore, $\epsilon_1^{\mathsf{T}}\tau(\gamma_n) = T_n^{\nu(n)}\gamma_n^{\mathsf{T}}$, and $\tau(\gamma_n) \neq 0$ iff $T_n^{\nu(n)}\gamma_n^{\mathsf{T}} \neq 0$.

We now state and prove our main result. First, however, we remind the reader of the definitions of the Segre and Weyr characteristics of a matrix. (See, for example, [3, pp. 152–154].) Specifically, let λ be an eigenvalue of A. The Weyr characteristic of A associated with λ is the (finite, nonincreasing) list of positive differences given by $\operatorname{rank}(A-\lambda I)^{j-1} - \operatorname{rank}(A-\lambda I)^j$, $j = 1,2,\dots$. The Segre characteristic of A associated with λ is the conjugate partition of the Weyr characteristic, or equivalently, is the (nonincreasing) list of sizes of λ-blocks in the Jordan canonical form of A.

THEOREM 2. *Let $\epsilon_1 = (1,0,\dots,0)$ and $\gamma_n = (c_1, c_2, \dots, c_n)$ be vectors in $\mathbb{C}^n$, and let $\tau(\gamma_n)$ be given by* (0.3) *above. Let $C_n = \gamma_n^{\mathsf{T}}\epsilon_1$, $D_n = (d_{ij})$ with $d_{ij} = 1$ if $i \mid j$ and* 0 *otherwise, and let $A_n = C_n + D_n$ be $n\times n$ complex matrices. Let*

$$\mathscr{S}_1(A_n) = \big(s_1(n), s_2(n), \dots, s_{w_1(n)}(n)\big)$$

be the Segre characteristic, and

$$\mathscr{W}_1(A_n) = \big(w_1(n), w_2(n), \dots, w_{s_1(n)}(n)\big)$$

be the Weyr characteristic of A_n associated with the eigenvalue 1. *If $\tau(\gamma_n) \neq 0$, then*

$$s_i(n) = \left[\log_2 \frac{n}{2i+1}\right] + 1, \qquad w_j(n) = \left[\frac{n}{2^{j-1}}\right] - \left[\frac{n}{2^j}\right] - 1.$$

Proof. By definition of the Weyr characteristic and by Theorem 1, since $T_n^{\nu(n)}\gamma_n^{\mathrm{T}} = \epsilon_1^{\mathrm{T}}\tau(\gamma_n) \neq 0$,

$$\begin{aligned} w_j(n) &= \operatorname{rank}(A_n - I_n)^{j-1} - \operatorname{rank}(A_n - I_n)^j \\ &= \left((j-1) + \left[\frac{n}{2^{j-1}}\right]\right) - \left(j + \left[\frac{n}{2^j}\right]\right) \\ &= \left[\frac{n}{2^{j-1}}\right] - \left[\frac{n}{2^j}\right] - 1 \end{aligned}$$

for $j = 1,2,\dots, s_1(n) = \nu(n/3)+1 = [\log_2(n/3)]+1$. In particular, if $n<3$, then $\mathscr{W}_1(A_n)$ is empty.

Next, we determine the conjugate partition of the Weyr characteristic, which provides the Segre characteristic $\mathscr{S}_1(A_n)$.

First, given j, let $t = [n/2^j]$. Since $t \times 2^j \leqslant n < (t+1)\times 2^j$, then t is the number of positive multiplies of 2^j which are at most n. Similarly, $[n/2^{j-1}]$ is the number of positive multiples of 2^{j-1} which are at most n. Since the even multiples of 2^{j-1} of at most n are in one-to-one correspondence with the multiples of 2^j of at most n, then $[n/2^{j-1}] - [n/2^j]$ is the number of odd multiples of 2^{j-1} of at most n.

Now, (i, j) is a point of the Segre-Weyr array of A_n associated with the eigenvalue 1 (see, for example, [3, p. 153]) iff $i \leqslant w_j(n)$ iff $i+1 \leqslant w_j(n)+1$ iff $i+1 \leqslant [n/2^{j-1}] - [n/2^j]$ iff the $(i+1)$st odd multiple of 2^{j-1} is at most n iff $(2i+1)\times 2^{j-1} \leqslant n$ iff $2^{j-1} \leqslant n/(2i+1)$ iff $j-1 \leqslant \log_2(n/(2i+1))$ iff $j \leqslant \log_2(n/(2i+1))+1$. If i is fixed, then $s_i(n)$ is the greatest integer j for which this last inequality is satisfied. That is, for $1 \leqslant i \leqslant w_1(n) = n - [n/2] - 1$,

$$s_i(n) = \left[\log_2 \frac{n}{2i+1}\right] + 1. \qquad \blacksquare$$

We express appreciation to Rodney W. Forcade and R. Vencil Skarda for their assistance in establishing the preceding conjugate partition.

COROLLARY 2. *Let the conditions be as in Theorem 2. If $\tau(\gamma_n) \neq 0$, then A_n is similar to a direct sum of a cyclic matrix of order $\nu(n)+1$ and $[(n-1)/2]$ Jordan 1-blocks of orders*

$$\left[\log_2\frac{n}{3}\right]+1, \left[\log_2\frac{n}{5}\right]+1, \dots, \left[\log_2\frac{n}{\{n\}}\right]+1,$$

where $\{n\} = 2[(n-1)/2]+1$; *in particular, the minimum polynomial of* A_n *is of degree*

$$[\log_2 n] + \left[\log_2 \frac{n}{3}\right] + 2,$$

and $\sigma_1(n)$, *the multiplicity of* 1 *in the characteristic polynomial of* A_n, *is* $n - \nu(n) - 1$.

Proof. Since each positive integer $k \leqslant n$ is uniquely identified with a pair (i, m) of nonnegative integers under the correspondence $k = (2i+1)\times 2^m$, where $1 \leqslant 2^m = k/(2i+1) \leqslant n/(2i+1)$ requires $0 \leqslant m \leqslant [\log_2(n/(2i+1))]$ and $1 \leqslant 2i+1 = k/2^m \leqslant n$ requires $0 \leqslant i \leqslant (n-1)/2$ and hence $0 \leqslant i \leqslant [(n-1)/2]$, we have

$$n = \sum_{i=0}^{[(n-1)/2]} \left(\left[\log_2 \frac{n}{2i+1}\right] + 1 \right).$$

Now, let $h_n(x)(x-1)^{\sigma_1(n)}$ be the characteristic polynomial of A_n. By Theorem 2,

$$\sigma_1(n) = \sum_{i=1}^{w_1(n)} s_i(n) = \sum_{i=1}^{n-[n/2]-1} \left(\left[\log_2 \frac{n}{2i+1}\right] + 1 \right).$$

Thus, since $[(n-1)/2] = n - [n/2] - 1$, the multiplicity $\sigma_1(n)$ of 1 in the characteristic polynomial of A_n is $n - ([\log_2 n]+1) = n - \nu(n) - 1$. (This result was also obtained in [1] for the case $\gamma_n = (0,1,\dots,1)$, $n > 1$.)

Next, let $\lambda \neq 1$ be an eigenvalue of A_n. Since the $(n-1)\times(n-1)$ lower right-hand block of $A_n - \lambda I_n$ is upper triangular with the nonzero element $1-\lambda$ on the diagonal, then $\operatorname{rank}(A_n - \lambda I_n) = n-1$ and there is precisely one Jordan λ-block in the Jordan normal form of A_n. The direct sum of these distinct Jordan λ-blocks for $\lambda \neq 1$ forms a cyclic matrix of order $\nu(n)+1$. Consequently, A_n is similar to a direct sum of a cyclic matrix of order $\nu(n)+1$ and $[(n-1)/2] = n - [n/2] - 1 = w_1(n)$ Jordan 1-blocks of orders $[\log_2(n/k)]+1$, $k = 3, 5, \dots, \{n\} = 2[(n-1)/2]+1$.

Finally, since the degree of the minimum polynomial of A_n is the sum of the orders of the largest Jordan blocks associated with the distinct eigenvalues of A_n, then the degree of the minimum polynomial of A_n is

$$([\log_2 n]+1) + ([\log_2(n/3)]+1). \qquad \blacksquare$$

We next show that if $\tau(\gamma_n)=0$, then the Jordan structure of A_n is different from the structure as given in Corollary 2.

LEMMA 3. *Let* $A_n=\gamma_n^{\mathsf{T}}\epsilon_1+I_n+T_n$ *be given as above. If* $T_n^{\nu(n)}\gamma_n^{\mathsf{T}}=0$, *then* $\operatorname{rank}(A_n-I_n)^{\nu(n)+1}<\nu(n)+1$.

Proof. By the proof of Theorem 1, $(A_n-I_n)^{\nu(n)+1}$ is the product of the $n\times(\nu(n)+1)$ matrix whose columns are

$$(C_n+T_n)^0\gamma_n^{\mathsf{T}},(C_n+T_n)^1\gamma_n^{\mathsf{T}},\ldots,(C_n+T_n)^{\nu(n)}\gamma_n^{\mathsf{T}}$$

and the $(\nu(n)+1)\times n$ matrix whose rows are

$$\epsilon_1T_n^{\nu(n)},\ldots,\epsilon_1T_n,\epsilon_1.$$

If $T_n^{\nu(n)}\gamma_n^{\mathsf{T}}=0$, then, by (2.1) of Lemma 2, the columns of the first factor are linearly dependent, and it follows that $\operatorname{rank}(A_n-I_n)^{\nu(n)+1}<\nu(n)+1$. ■

COROLLARY 3. *If* $\sigma_1(n)$ *is the multiplicity of* 1 *in the characteristic polynomial of* A_n, *then* $\sigma_1(n)\geqslant n-\nu(n)-1$.

Proof. By Theorem 1 and Lemma 3,

$$\operatorname{rank}(A_n-I_n)^{\nu(n)+1}\leqslant\nu(n)+1.$$

This means that $\sigma_1(n)\geqslant n-(\nu(n)+1)$. ■

THEOREM 3. *Let the notation be as in Theorem 2, and let* $\sigma_1(n)$ *be the multiplicity of* 1 *in the characteristic polynomial of* A_n. *Then the following statements are equivalent*:

(3.1) $\tau(\gamma_n)\neq0$.
(3.2) $s_i(n)=[\log_2(n/(2i+1))]+1,\ i=1,2,\ldots,[(n-1)/2]$.
(3.3) $\sigma_1(n)=n-\nu(n)-1$.
(3.4) $\operatorname{rank}(A_n-I_n)^{\nu(n)+1}=\nu(n)+1$.

Proof. (3.1) $\to$ (3.2): This is the statement of Theorem 2.

(3.2) $\to$ (3.3): Since $\sigma_1(n)$ is the sum of the entries in the Segre characteristic of A_n associated with 1, then, by the first part of the proof of Corollary 2, given (3.2), $\sigma_1(n)=n-\nu(n)-1$.

(3.3) → (3.4): Let $\sigma_1(n) = n - \nu(n) - 1$. Since the rank of any power of $A_n - I_n$ is at least $n - \sigma_1(n)$, then in particular

$$\operatorname{rank}(A_n - I_n)^{\nu(n)+1} \geqslant n - \sigma_1(n) = \nu(n) + 1.$$

But, by the proof of Corollary 3, the reverse inequality holds; hence, (3.4) is satisfied.

(3.4) → (3.1): Given (3.4), by Lemma 3, $T_n^{\nu(n)}\gamma_n^{\mathsf{T}} \neq 0$, which is equivalent to $\tau(\gamma_n) \neq 0$. ■

The matrix A_n defines a linear transformation

$$\mathscr{A}_n : \mathbf{C}^n \to \mathbf{C}^n,\ \zeta \to \zeta A_n$$

on the vector space $\mathbf{C}^n$ of row vectors of length n over $\mathbf{C}$. We conclude this paper by providing a basis of $\mathbf{C}^n$ such that the matrix of $\mathscr{A}_n$ with respect to this basis is the direct sum of matrices given in Corollary 2.

LEMMA 4. *Let the conditions be as in Theorem* 2, *and suppose that* $\tau(\gamma_n) \neq 0$. *Let* E_n *be the primary idempotent of* A_n *associated with the eigenvalue* 1. *Then* $\epsilon_1 E_n = 0$, $E_n\gamma_n^{\mathsf{T}} = 0$, $T_nE_n = E_nT_n$, *and*

$$\epsilon_1, \epsilon_1 A_n, \dots, \epsilon_1 A_n^{\nu(n)},$$

$$\epsilon_3 E_n, \epsilon_3 E_n T_n, \dots, \epsilon_3 E_n T_n^{\nu(n/3)},$$

$$\dots,$$

$$\epsilon_{\{n\}} E_n, \epsilon_{\{n\}} E_n T_n, \dots, \epsilon_{\{n\}} E_n T_n^{\nu(n/\{n\})},$$

where $\{n\} = 2[(n-1)/2] + 1$, *is a basis of* $\mathbf{C}^n$.

Proof. Let A_n be given as above with the assumption that $\tau(\gamma_n) \neq 0$. Let E_n be the primary idempotent of A_n associated with the eigenvalue 1. (See, for example, [2, p. 132].) Since $s_1(n)$ is the degree of the factor $x - 1$ in the minimum polynomial of A_n, then $I_n - E_n = g_n(A_n)(A_n - I_n)^{s_1(n)}$ for some polynomial $g_n(x)$. In particular, the row space of $I_n - E_n$ is contained in the row space of $(A_n - I_n)^{s_1(n)}$. But, by Theorem 1, $\operatorname{rank}(A_n - I_n)^{s_1(n)} = \operatorname{rank}(A_n - I_n)^{\nu(n)+1}$. Therefore, the row space of $(A_n - I_n)^{s_1(n)}$ is the same as the row space of $(A_n - I_n)^{\nu(n)+1}$, which by the proof of Lemma 3 is contained in the space spanned by the vectors $\epsilon_1, \epsilon_1 T_n, \dots, \epsilon_1 T_n^{\nu(n)}$. Since

$\operatorname{rank}(I_n - E_n)$ is $\nu(n)+1$, the sum of the multiplicities of the eigenvalues of A_n that are not equal to 1, it follows that $\epsilon_1, \epsilon_1 T_n, \dots, \epsilon_1 T_n^{\nu(n)}$ is in fact a basis of the row space of $I_n - E_n$. In particular, $\epsilon_1 = \alpha(I_n - E_n)$ for some row vector $\alpha \in \mathbb{C}^n$. Since E_n is idempotent, $\epsilon_1 E_n = 0$.

By an analogous argument, the column space of $I_n - E_n$ is the space spanned by the columns

$$\gamma_n^{\mathsf{T}}, (C_n + T_n)\gamma_n^{\mathsf{T}}, \dots, (C_n + T_n)^{\nu(n)}\gamma_n^{\mathsf{T}}.$$

In particular, $\gamma_n^{\mathsf{T}} = (I_n - E_n)\beta^{\mathsf{T}}$ for some $\beta \in \mathbb{C}^n$; hence, $E_n\gamma_n^{\mathsf{T}} = 0$.

Therefore, since $T_n = A_n - I_n - \gamma_n^{\mathsf{T}}\epsilon_1$, E_n is a polynomial in A_n, $\epsilon_1 E_n = 0$, and $E_n\gamma_n^{\mathsf{T}} = 0$, then

$$\begin{aligned} E_n T_n &= E_n\left(A_n - I_n - \gamma_n^{\mathsf{T}}\epsilon_1\right) = E_n A_n - E_n \\ &= AE_n - E_n = \left(A_n - I_n - \gamma_n^{\mathsf{T}}\epsilon_1\right)E_n = T_n E_n. \end{aligned}$$

Finally, consider the list of vectors given in the statement of Lemma 4. By Corollary 2 this list contains n elements. We show that this list is a basis of $\mathbb{C}^n$ by demonstrating that it is linearly independent. Thus, suppose

$$0 = \sum_{m=0}^{\nu(n)} a_{m,1}\epsilon_1 A_n^m + \sum_{i=1}^{[(n-1)/2]} \sum_{m=0}^{\nu(n/(2i+1))} a_{m,2i+1}\epsilon_{2i+1}E_n T_n^m.$$

Multiply this linear combination on the right by E_n. Since E_n commutes with both A_n and T_n, $\epsilon_1 E_n = 0$ and E_n is idempotent, then

$$0 = \left(\sum_{i=1}^{[(n-1)/2]} \sum_{m=0}^{\nu(n/(2i+1))} a_{m,2i+1}\epsilon_{2i+1}T_n^m \right) E_n.$$

But this requires the row space of the first factor of the product on the right to be in the row space of $I_n - E_n$. Therefore,

$$\sum_{i=1}^{[(n-1)/2]} \sum_{m=0}^{\nu(n/(2i+1))} a_{m,2i+1}\epsilon_{2i+1}T_n^m = \sum_{m=0}^{\nu(n)} b_m \epsilon_1 T_n^m.$$

Now, as in the proof of Corollary 2, each positive integer $k \leqslant n$ is uniquely identified with the pair (i, m) of nonnegative integers under the correspondence $k=(2i+1)2^m$ with $0 \leqslant m \leqslant [\log_2(n/(2i+1))] = \nu(n/(2i+1))$ and $0 \leqslant i \leqslant [(n-1)/2]$. Since, by Corollary 1, the first nonzero element of the row vector $\epsilon_{2i+1}T_n^m$ is a 1 in column $(2i+1)2^m$, it follows that $\{\epsilon_{2i+1}T_n^m\}$ is linearly independent; hence, $a_{m,2i+1}=0$ for $0 \leqslant m \leqslant \nu(n/(2i+1))$ and $1 \leqslant i \leqslant [(n-1)/2]$ [and $b_m = 0$ for $0 \leqslant m \leqslant \nu(n)$]. Finally, by (2.2) and (2.4) of Lemma 2, $a_{m,1}=0$ for $0 \leqslant m \leqslant \nu(n)$. ■

COROLLARY 4. *Let the conditions be as in Theorem 2, and suppose that $\tau(\gamma_n) \neq 0$. Then the matrix representation of the linear transformation $\mathscr{A}_n : \mathbf{C}^n \to \mathbf{C}^n$, $\zeta \to \zeta A_n$ with respect to the basis of Lemma 4 is the direct sum of the companion matrix of a polynomial $h_n(x)$ of degree $[\log_2 n]+1$ with $h_n(1) \neq 0$, and $[(n-1)/2]$ Jordan 1-blocks of orders*

$$\left[\log_2 \frac{n}{3}\right]+1, \left[\log_2 \frac{n}{5}\right]+1, \dots, \left[\log_2 \frac{n}{\{n\}}\right]+1,$$

where $\{n\} = 2[(n-1)/2]+1$.

Proof. First, let $1 \leqslant i \leqslant [(n-1)/2]$. Since $E_n\gamma_n^{\mathsf{T}} = 0$ and $T_nE_n = E_nT_n$, then

$$\begin{aligned}(\epsilon_{2i+1}E_nT_n^m)A_n &= \epsilon_{2i+1}E_nT_n^m(\gamma_n^{\mathsf{T}}\epsilon_1 + I_n + T_n)\\ &= \epsilon_{2i+1}E_nT_n^m + \epsilon_{2i+1}E_nT_n^{m+1}.\end{aligned}$$

Also, by (1.3) of Corollary 1,

$$\epsilon_{2i+1}E_nT_n^{\nu(n/(2i+1))+1} = \epsilon_{2i+1}T_n^{\nu(n/(2i+1))+1}E_n = 0.$$

Consequently, the matrix of the restriction of $\mathscr{A}_n$ to the $\mathscr{A}_n$-invariant subspace spanned by $\epsilon_{2i+1}E_n, \epsilon_{2i+1}E_nT_n, \dots, \epsilon_{2i+1}E_nT_n^{\nu(n/(2i+1))}$ is the Jordan 1-block

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

of order $\nu(n/(2i+1)+1=[\log_2(n/(2i+1))]+1$.

Next, by (2.2) of Lemma 2 and the fact that $T_n^{\nu(n)+1}=0$, $\epsilon_1 A_n^{\nu(n)+1}$ is a linear combination of $\epsilon_1, \epsilon_1 A_n, \dots, \epsilon_1 A_n^{\nu(n)}$; that is, $\epsilon_1 h_n(A_n)=0$ for some monic polynomial $h_n(x)$ of degree $\nu(n)+1$. The matrix of the restriction of $\mathscr{A}_n$ to the $\mathscr{A}_n$-cyclic subspace generated by ϵ_1 is the companion matrix of $h_n(x)$. Since the multiplicity of 1 in the characteristic polynomial of A_n is

$$\sigma_1(n)=n-\nu(n)-1=n-([\log_2 n]+1)=\sum_{i=1}^{[(n-1)/2]}\left(\left[\log_2\frac{n}{2i+1}\right]+1\right),$$

then the above Jordan 1-blocks account for the Jordan 1-structure of $\mathscr{A}_n$. Therefore, $x-1$ is not a factor of $h_n(x)$, and $h_n(1)\neq 0$. ∎

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